We write out the expressions for the velocity on the axis of the jet:

$$u_m/U = X^{2/8}[1 + (1/8 - 2a_4/a_0)X^{-4/8}],$$

and also for the velocity and friction on the separation boundary:

$$u_*/U = X^{2/8}[1 - (1/8 + 2a_4/a_0)X^{-4/8}],$$
$$\tau_* = -\mu_1(U/a_0)X^{-1/8}(1/2)[1 + (a_4/a_0 - A/4)X^{-4/8}].$$

In the conclusion, we point out that for an outflow of a heavy liquid downward the solutions written out will be valid for $\rho_1 > \rho_2$ and

$$B = g\frac{a_0^2}{\nu_1 U}\frac{\rho_1 - \rho_2}{\rho_1}.$$

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STABILIZATION OF SOLUTIONS OF TWO-DIMENSIONAL EQUATIONS OF DYNAMICS OF AN IDEAL LIQUID

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UDC 532.5 +517.9

Problems of solvability of initial- and boundary-value problems for two-dimensional nonstationary Euler equations of dynamics of an ideal liquid have been studied by many authors. A review and the corresponding references can be found, for example, in [1, 2]. However, the problem of asymptotic behavior of the solutions of the Euler equation as $t \to \infty$ has not been investigated.

This is apparently explained by the fact that the corresponding boundary-value problems for a stationary Euler equation do not possess the uniqueness property of the solution. In addition, examples exist where a stationary boundary-value problem has a continuum of solutions, as, for example, the problem with the condition of no leakage of the liquid through the boundary of a region of flow. To obtain any results about the asymptotic behavior in the case of $t \to \infty$ of the solutions of nonstationary initial-value problems, we have to single out a class in which the corresponding stationary problem has a unique solution (or a finite number of solutions). One such class was introduced in [3]. The simplest representative of this class is motion without vortices. In the present paper we present sufficient conditions under which the solutions of two-dimensional Euler equations as $t \to \infty$ tend to a potential flow.

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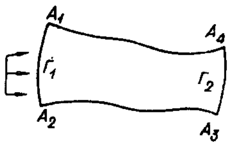

Fig. 1

1. Formulation of the Problem

Let Ω be a bounded simply connected region of the plane $x=(x_1, x_2)$ with a piecewise-smooth boundary Γ. For the sake of simplicity, we shall assume that the region Ω has the form of a curvilinear four-cornered figure A_1, A_2, A_3, A_4 with the smooth sides A_jA_{j+1}, $1 \le j \le 4$, $A_5 \equiv A_1$ (see Fig. 1), so that $\Gamma_0 = A_2A_3 \cup A_4A_1$, $\Gamma_1 = A_1A_2$, $\Gamma_2 = A_3A_4$, $\Gamma_3 = \bigcup_{j=1}^{4} A_j$, $\Gamma = \bigcup_{i=0}^{3} \Gamma_i$. By $\mathbf{n}=(n_1, n_2)$ we shall denote the vector of the inner normal to the boundary at the points $\Gamma \setminus \Gamma_3$. Let $I_T = \{t \in R \mid T \leqslant t < \infty\}$, where T is an arbitrary nonnegative number. The boundaries A_jA_{j+1} and the angles $\pi\beta_j$ at the vertices A_j will be assumed to satisfy the conditions

$$A_jA_{j+1} \in C^{2+\alpha},\ 0 < \alpha < 1;\ 0 < \beta_j \leqslant 1/2,\ 1 \leqslant j \leqslant 4. \qquad \text{I}$$

In the region $\Omega \times I_0$ we shall consider the initial-value problem for two-dimensional nonstationary Euler equations

$$\begin{gathered} \mathbf{v}' + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p,\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0(x),\ \mathbf{v}\cdot\mathbf{n}|_\Gamma = \gamma(x, t),\ \omega|_{\Gamma_1} = 0, \end{gathered} \qquad (1.1)$$

where $\mathbf{v} = (v_1, v_2)$ is the velocity; p is the pressure; and $\boldsymbol{\omega} = \operatorname{rot}\mathbf{v}$ is the vorticity. We denote the contraction of the function $\gamma(x, t)$ on Γ_i by $\gamma^{(i)}(x, t)$ $(i=0, 1, 2)$ and assume that the conditions

$$\mathbf{v}_0 \in C^1(\bar\Omega),\ \operatorname{div}\mathbf{v}_0 = 0,\ \omega_0|_{\Gamma_1} = 0; \qquad \text{II}$$

$$\begin{gathered} \gamma^{(0)} \equiv 0,\ \gamma^{(1)} > 0,\ \gamma^{(2)} < 0;\ \int_\Gamma \gamma(x, t)\, d\sigma = 0,\ t \geqslant 0; \\ \gamma^{(i)} \in C^1(\Gamma_i \times I_0),\ i = 1,2;\ \gamma(x, 0) = \mathbf{v}_0\cdot\mathbf{n},\ x \in \Gamma\setminus\Gamma_3 \end{gathered} \qquad \text{III}$$

are fulfilled, where $\boldsymbol{\omega}_0 = \operatorname{rot}\mathbf{v}_0$; $d\sigma$ is an element of length of an arc of the boundary Γ; and $\bar\Omega$ is the region Ω at the instant $t=0$. In addition, we shall assume that there exists a vector $\tilde{\mathbf{u}}(x, t) \in C^1(\bar\Omega \times I_0)$ such that $\tilde{\mathbf{u}}\cdot\mathbf{n}|_\Gamma = \gamma(x, t)$, $t \geqslant 0$, and $\gamma(x, t)$ uniformly tends to the function $\gamma_\infty(x)$ as $t \to \infty$, with the contractions $\gamma_\infty^{(i)}(x)$ of the function $\gamma_\infty(x)$ on Γ_i $(i=0, 1, 2)$ satisfying the conditions

$$\begin{gathered} \gamma_\infty^{(0)} \equiv 0,\ \gamma_\infty^{(1)} \geqslant \varepsilon,\ \gamma_\infty^{(2)} \leqslant -\varepsilon, \\ \gamma_\infty^{(i)} \in C^1(\Gamma_i),\ i = 1, 2, \end{gathered} \qquad \text{IV}$$

where $\varepsilon = \text{const} > 0$.

Side by side with the problem (1.1), we shall also consider the corresponding boundary-value problem for the stationary Euler equations

$$\begin{gathered} \mathbf{v}\cdot\nabla\mathbf{v} = -\Delta p,\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}\cdot\mathbf{n}|_\Gamma = \gamma_\infty(x),\ \omega|_{\Gamma_1} = 0. \end{gathered} \qquad (1.2)$$

It is obvious that the potential flow $\mathbf{u}_\infty(x)$ given by the expressions

$$\operatorname{rot}\mathbf{u}_\infty = 0,\ \operatorname{div}\mathbf{u}_\infty = 0,\ \mathbf{u}_\infty\cdot\mathbf{n}|_\Gamma = \gamma_\infty(x), \qquad (1.3)$$

together with the pressure $p_\infty(x) = \text{const} - \mathbf{u}_\infty^2(x)/2$ is the solution of the problem (1.2). This solution is unique in the class of functions introduced in [3], for which $\sup_{x\in\Omega} |\boldsymbol{\omega}(x)| \leqslant \sup_{x\in\Gamma_1} |\boldsymbol{\omega}(x)|$.

We shall find the conditions under which the solution of the nonstationary problem (1.1) tends, as $t \to \infty$, to the stationary potential flow $\mathbf{u}_\infty(x)$.

To solve this problem we shall first indicate the conditions under which any trajectory of the vector field $\mathbf{v}(x, t)$, commencing at $t=0$ in the region Ω, leaves Ω in a finite time, while subsequently we shall use the property of conservation of vorticity along the trajectories of the field $\mathbf{v}$, a property which is known for the solutions of the problem (1.1).

In the following we shall consider only vector fields which satisfy the condition div $\mathbf{v}=0$. Following the terminology of [4], we shall call any such vector field a flow. If in addition to the condition div $\mathbf{v}=0$ the condition $\mathbf{v}\cdot\mathbf{n}|_\Gamma=0$ is fulfilled, then the field $\mathbf{v}$ is called a tangential flow. We note that the solution $\mathbf{v}(x,t)$ of the problem (1.1) is conveniently considered as the sum of two flows: $\mathbf{v}(x,t)=\mathbf{u}(x,t)+\mathbf{w}(x,t)$, where $\mathbf{u}$ is a nonstationary potential flow given by the expressions

$$\operatorname{rot}\mathbf{u}=0,\ \operatorname{div}\mathbf{u}=0,\ \mathbf{u}\cdot\mathbf{n}|_\Gamma=\gamma(x,t), \tag{1.4}$$

while $\mathbf{w}$ is a tangential vortical flow which for $t\geq 0$ is the solution of the problem

$$\operatorname{rot}\mathbf{w}=\omega(x,t),\ \operatorname{div}\mathbf{w}=0,\ \mathbf{w}\cdot\mathbf{n}|_\Gamma=0, \tag{1.5}$$

where $\omega(x,t)=\operatorname{rot}\mathbf{v}$. In view of the assumptions made about the functions $\gamma(x,t)$ and $\gamma_\infty(x)$, obviously $\mathbf{u}\in C(\overline{\Omega}\times I_0)$, $\mathbf{u}_\infty\in C(\overline{\Omega})$, with

$$\|\mathbf{u}(x,t)-\mathbf{u}_\infty(x)\|_{C(\overline{\Omega})}\to 0,\ t\to\infty. \tag{1.6}$$

Below, under the vectors $\mathbf{u}_\infty$, $\mathbf{u}$, and $\mathbf{w}$ we shall understand flows defined, respectively, by the expressions (1.3)-(1.5).

2. The Existence of a Generalized Solution

We shall prove the existence of a solution of the problem (1.1) and indicate some of its properties. We put

$$Q=\Omega\times(0,T),\ \Sigma=\Gamma\times(0,T),\ \Sigma_i=\Gamma_i\times(0,T),\ i=0,1,2,$$

where T is a sufficiently large positive number. We denote by V the space of tangential flows $\mathbf{v}\in H^1(\Omega)$; by $\mathbf{H}$ we denote the closure of elements of the space $\mathbf{V}$ in the norm $H^\prime(\Omega)$. We also introduce a set Φ of "trial" functions $\varphi(x,t)\in C^1(\overline{Q})$, which are zero on Σ_2 and for $t=T$.

Definition 2.1. A flow $\mathbf{v}\equiv\mathbf{u}+\mathbf{w}$ is called the generalized solution of the system (1.1), if $\mathbf{w}\in L^\infty(0,T;\mathbf{V})$, $\mathbf{w}'\in L^2(0,T;\mathbf{H})$, $\operatorname{rot}\mathbf{w}\in L^\infty(Q)$, $\mathbf{v}(x,0)=\mathbf{v}_0(x)$, and for any function $\varphi\in\Phi$ the integral identity

$$F(\mathbf{v},\varphi)\equiv\int_Q\int\operatorname{rot}\mathbf{v}\,(\varphi'+\mathbf{v}\cdot\nabla\varphi)\,dx\,dt+\int_\Omega\omega_0(x)\,\varphi(x,0)\,dx=0 \tag{2.1}$$

is fulfilled.

The existence of a solution of the problem (1.1) can be shown by the vanishing viscosity method. For this we assume first that $\omega_0(x)\in C^2(\overline{\Omega})$, and consider the initial-value problem for the Navier–Stokes equations, which in terms of the variables $\mathbf{v}$, ω has the form

$$\begin{gathered}\omega'+\mathbf{v}\cdot\nabla\omega-\nu\Delta\omega=0,\ \operatorname{rot}\mathbf{v}=\omega,\ \operatorname{div}\mathbf{v}=0,\\ \omega|_{t=0}=\omega_0(x),\ \omega|_\Sigma=0,\ \mathbf{v}\cdot\mathbf{n}|_\Sigma=\gamma(x,t).\end{gathered} \tag{2.2}$$

Reasoning in the same way as in [1] in the case of a smooth region, we can show that for each $\nu>0$ the problem (2.2) has a unique solution $\mathbf{v}_\nu\equiv\mathbf{u}+\mathbf{w}_\nu$, where for the functions $\mathbf{w}_\nu$, $\omega_\nu\equiv\operatorname{rot}\mathbf{w}_\nu$ the estimates

$$\begin{gathered}\|\omega_\nu\|_{L^\infty(Q)}\leqslant M_0=\|\omega_0(x)\|_{C(\overline{\Omega})},\\ \|\mathbf{w}_\nu\|_{L^\infty(0,T;\mathbf{V})}\leqslant M_1,\ \|\mathbf{w}_\nu'\|_{L^2(0,T;\mathbf{H})}\leqslant M_2,\\ \nu\int_0^T\|\nabla\omega_\nu\|^2_{L^2(\Omega)}\,dt\leqslant M_3\end{gathered} \tag{2.3}$$

are fulfilled. Here the constants M_j $(j=0,\dots,3)$ do not depend on ν. By virtue of the estimates (2.3) from the family of functions $\mathbf{w}_\nu$, ω_ν we can extract sequences $\mathbf{w}_k\equiv\mathbf{w}_{\nu_k}$, $\omega_k\equiv\omega_{\nu_k}$, such that for $\nu_k\to 0$ $(k\to\infty)$ $\mathbf{w}_k\to\mathbf{w}$ strongly in $L^2(Q)$, $\omega_k\to\omega$ weakly in $L^\infty(Q)$, with $\mathbf{w}\in L^\infty(0,T;\mathbf{V})$, $\mathbf{w}'\in L^2(0,T;\mathbf{H})$, $\operatorname{rot}\mathbf{w}=\omega$.

Let $\sigma>0$ be an arbitrary sufficiently small number. We denote by φ_σ a function from the class $C^2(\overline{\Omega})$, which is equal to zero in σ, the neighborhood of the set Γ_3, and equal to unity outside 2σ, the neighborhood Γ_3. We multiply the equation for ω_k by the product $\varphi_\sigma\varphi$, where $\varphi\in\Phi$, and integrate the resulting expression by parts. We have

$$\int\int\omega_k[\varphi_\sigma\varphi'+\mathbf{v}_k\cdot\nabla(\varphi_\sigma\varphi)]\,dxdt+\int_\Omega\omega_0(x)\,\varphi_\sigma(x)\,\varphi(x,0)\,dx=\nu_k\int_{\Sigma_0\cup\Sigma_1}\int\frac{\partial\omega_k}{\partial n}\varphi_\sigma\varphi\,d\sigma dt+\nu_k\int_Q\int\nabla\omega_k\cdot\nabla(\varphi_\sigma\varphi)\,dxdt. \tag{2.4}$$

The second term on the right side of (2.4), in view of the last estimate in (2.3), tends to zero as $\nu_k \to 0$. The tending to zero of the first term as $\nu_k \to 0$ can be shown with the use of Lemma 4.1 of [1], which is valid at regular points of the boundary Γ. Going to the limit in (2.4) as $\nu_k \to 0 (k \to \infty)$, we find that the flow $\mathbf{v} \equiv \mathbf{u} + \mathbf{w}$ satisfies the integral identity

$$F(\mathbf{v}, \varphi_\sigma\varphi) \equiv \int_Q\int \omega [\varphi_\sigma\varphi' + \mathbf{v} \cdot \nabla (\varphi_\sigma\varphi)]\, dxdt + \int_\Omega \omega_0(x)\, \varphi_\sigma(x)\, \varphi(x, 0)\, dx = 0. \tag{2.5}$$

We shall show that the flow $\mathbf{v}$ satisfies the integral identity (2.1). For this it is sufficient to show that the identity (2.5) remains valid if we put $\sigma = 0$ in it. We consider the difference

$$F(\mathbf{v}, \varphi_\sigma\varphi) - F(\mathbf{v}, \varphi) \equiv \int_Q\int \omega (\varphi_\sigma - 1)(\varphi' + \mathbf{v} \cdot \nabla\varphi)\, dxdt + \\ + \int_\Omega \omega_0(x)(\varphi_\sigma - 1)\, \varphi(x, 0)\, dx + \int_Q\int \omega\varphi\mathbf{v} \cdot \nabla\varphi_\sigma dxdt = 0. \tag{2.6}$$

The first two terms in (2.6) obviously tend to zero as $\sigma \to 0$. We consider the last term on the right side of (2.6). Taking into account the fact that $|\nabla\varphi_\sigma| = 0(\sigma^{-1})$, and applying the Hölder inequality, we have

$$\left|\int_Q\int \omega\varphi\mathbf{v} \cdot \nabla\varphi_\sigma dxdt\right| = \left|\int_0^T \int_{U_{2\sigma}} \omega\varphi\mathbf{v} \cdot \nabla\varphi_\sigma dxdt\right| \leqslant$$

$$\leqslant \left[\int_0^T \int_{U_{2\sigma}} (\omega\mathbf{v})^2\, dx\right]^{1/2} \left[\int_0^T \int_{U_{2\sigma}} \varphi^2 (\nabla\varphi_\sigma)^2\, dxdt\right]^{1/2} \leqslant \text{const} \int_0^T \int_{U_{2\sigma}} (\omega\mathbf{v})^2\, dxdt,$$

where $U_{2\sigma} \equiv U_{2\sigma}(\Gamma_3) - 2\sigma$, the neighborhood Γ_3. In view of the properties of the functions $\mathbf{w}$, ω it follows from this inequality that the last term in (2.6) tends to zero as $\sigma \to 0$. This means that $\mathbf{v}$ satisfies the integral identity (2.1) for any function $\varphi \in \Phi$ and consequently, is the generalized solution of the problem (1.1) in the sense of the Definition 2.1.

Thus, the existence of a solution of the problem (1.1) is proved for a smooth function $\omega_0(x)$. In the general case, when $\mathbf{v}_0(x) \in C^1(\bar{\Omega})$ and, consequently, $\omega_0(x) \in C(\bar{\Omega})$, the existence of a solution of the problem (1.1) is proved by means of an approximation of the function $\omega_0(x)$ by a sequence of sufficiently smooth functions and subsequently going to a limit.

We shall now establish certain properties of the generalized solution. First of all, in view of $\text{rot}\, \mathbf{w} \in L^\infty(Q)$ and the condition I, we can show, using the results of [5], that $\nabla \mathbf{w} \in L^\infty(0, T; L^q(\Omega))$, where $q > 2$ is an arbitrary number. From the Euler equations, just as in [1], we find that $\mathbf{w}' \in L^q(Q)$ and, consequently, by enclosure theorems, that $\mathbf{w} \in C^\theta(\bar{Q})$, where $\theta < 1$ is an arbitrary positive number.

In addition, according to [5], the flow $\mathbf{v} \in \mathbf{u} + \mathbf{w}$ satisfies in the neighborhood $U(x_0, t_0)$ of each point $(x_0, t_0) \in \bar{Q}$ the Lipschitz quasicondition

$$|\mathbf{v}(x, t) - \mathbf{v}(y, t)| \leqslant K_0 |x - y|(1 + |\ln|x - y||), \tag{2.7}$$

where $(x, t), (y, t) \in U(x_0, t_0)$ are arbitrary points; K_0 is a constant that does not depend on x, y, t. The condition (2.7), as we know, ensures unique solvability in a certain neighborhood of the point (x, t) of the problem

$$y' = \mathbf{v}(y, \tau), \; y|_{\tau = t} = x, \tag{2.8}$$

of the solution $y = y(x, t, \tau)$, which we shall call trajectories of the flow $\mathbf{v}$. Consequently, through each point $(x, t) \in \bar{Q}$ there passes a single trajectory of the flow $\mathbf{v}$. Using the results of the theory of dynamic systems, we can show that each trajectory of the flow $\mathbf{v}$ begins on $\bar{\Sigma}_1 \cup \bar{\Omega}_0$ and ends on $\bar{\Sigma}_2 \cup \bar{\Omega}_T$. Here by Ω_0 and Ω_T we have denoted the lower and upper bases of the cylinder Q. It can be shown that the function $y(x, t, \tau)$ is continuous (according to Hölder) with respect to x, t and continuously differentiable with respect to τ everywhere in $\bar{Q} \times [Q, T]$.

We introduce for each point $(x, t) \in \bar{Q}$ the functions $\tau_0(x, t)$ and $y_0(x, t)$, representing time and the point of entry of the trajectory $y(x, t, \tau)$ into the region Ω. Reasoning in the same way as in [1] in the proof of Lemmas 6.1 and 6.2, we can show that $\tau_0, y_0 \in C(\bar{Q})$ and the function $\omega \equiv \text{rot}\, \mathbf{v}$ has the representation

$$\omega(x, t) = \begin{cases} 0, & \tau_0(x, t) > 0; \\ \omega_0(y_0(x, t)), & \tau_0(x, t) = 0. \end{cases} \tag{2.9}$$

From (2.9) and the conditions II, in particular, it follows that $\omega(x, t) \in C(\bar{Q})$. As a result, we arrive at the following theorem.

rep-

THEOREM 2.1. Let the conditions I-III be fulfilled. Then there exists a generalized solution $\mathbf{v}(x, t)$ of the problem (1.1), with $\mathbf{v} \in C^{\theta}(\bar{Q})$, $0 < \theta < 1$, while the function $\boldsymbol{\omega} \equiv \operatorname{rot} \mathbf{v}$ is continuous in $\bar{Q}$ and has the re-resentation (2.9).

Remark 2.1. If we assume that the initial and boundary functions are sufficiently smooth, then, using (2.9) and the methods of [1], we can show that the generalized solution indeed is classical and unique.

We introduce the notation

$$\|\omega_0(x)\|_{C(\bar{\Omega})} = \lambda.$$

In the following the number λ is considered to be a parameter which varies between the limits $0 \le \lambda \le 1$.

3. Asymptotic Properties of the Trajectories

We shall study the asymptotic properties of the solutions of the problem (2.8) which determine the trajectories $y(x, t, \tau)$ of the flow $\mathbf{v}$. From the results of Secs. 1 and 2 it follows that $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u}$ is a solution of the problem (1.4), while $\mathbf{w} \in C^{\theta}(\bar{Q})$, with

$$\|\mathbf{w}\|_{C^{\theta}(\bar{Q})} \leqslant K_1\lambda. \tag{3.1}$$

Here K_1 is a certain constant depending on θ.

We agree on the following terminology. If at a certain time instant $\tau_1 \geqslant 0$ $y_1 \equiv y(x, t, \tau_1) \in \bar{\Gamma}_1$ (or $\bar{\Gamma}_2$), then we shall say that the trajectory $y(x, t, \tau)$ at the instant τ_1 enters into the region Ω (respectively, comes out of Ω), while the point (y_1, τ_1) will be called the point of entry (the point of exit) of the given trajectory in relation to the region Ω. If at all time instants $\tau \ge t$ the trajectory $y(x, t, \tau)$ is completely contained in the region $\Omega \cup \Gamma_0$, then such a trajectory is said to be asymptotic in Ω.

We shall consider in Ω the autonomous system

$$x' = \mathbf{u}_{\infty}(x). \tag{3.2}$$

The solutions of the system (3.2) are called, as we know, the streamlines of the vector field $\mathbf{u}_{\infty}(x)$ in the region Ω. Using the condition IV, we can show that at all points of $\bar{\Omega}$, $\mathbf{u}_{\infty} \neq 0$ [3]. Hence, it is easy to conclude that through each point $x \in \bar{\Omega}$ there passes a single streamline, with each streamline entering into Ω through the part $\bar{\Gamma}_1$ and coming out of Ω through the part $\bar{\Gamma}_2$. Since the streamlines coincide with the trajectories of the flow $\mathbf{u}_{\infty}$, this fact signifies that in the region Ω there are no asymptotic trajectories of the flow $\mathbf{u}_{\infty}$. An analogous fact holds for a nonstationary flow $\mathbf{v}(x, t)$

We denote by t_0 the value of the time t, starting from which the relation

$$|\gamma(x, t) - \gamma_{\infty}(x)| \leqslant \varepsilon/2, \; x \in \Gamma_1 \cup \Gamma_2, \; t \geqslant t_0$$

is fulfilled; here ε is the same as in IV. Such a value t_0 exists in view of the uniform convergence of γ to γ_{∞}. For any set D, by $U_{\delta}(D)$ we shall denote the neighborhood δ of the set D relative to $\bar{\Omega}$, i.e., a set of points $x \in \bar{\Omega}$, satisfying the condition dist $(x, D) < \delta$. We put $\Omega_{\delta} = \Omega \setminus U_{\delta}(\Gamma_1 \cup \Gamma_2)$.

LEMMA 3.1. There exist numbers $\delta = \delta(\varepsilon)$ and $s = s(\varepsilon)$ such that in the region $U_{\delta}(\Gamma_i)$ there are no asymptotic trajectories of the flow $\mathbf{v}$ for $\lambda \in [0, 1]$, with any trajectory from the initial point in $U_{\delta}(\Gamma_i) \times I_{t_0}$ exiting from $U_{\delta}(\Gamma_i)$ $(i = 1, 2)$ within a time interval $\Delta t \le s$.

Proof. We shall first consider the part Γ_2. We shall assume for the sake of simplicity that Γ_2 is a segment of the axis x_2, the region Ω lies in the half-plane $x_1 < 0$, and in a certain neighborhood of the corner points A_3 and A_4 the nonpenetrable parts of the boundary Γ are segments of straight lines. If this is not so, then it is first necessary to map a certain neighborhood of Γ_2 relative to Ω onto a region having a boundary of this type (for example, by means of conformal mapping) and, subsequently, carry out analogous considerations.

According to the assumption just introduced and in view of (1.4), (1.5) we have $w_1|_{\Gamma_2} \equiv 0$, $u_1|_{\Gamma_2} = -\gamma(x, t)$. From the properties of the flow $\mathbf{u}$ and the relations (3.1) it follows that there exists a neighborhood $U_{\delta}(\Gamma_2)$ such that for $(x, t) \in U_{\delta}(\Gamma_2) \times I_{t_0}$ $|w_1(x, t)| \leqslant \varepsilon/8$, $u_1(x, t) \ge \varepsilon/4$ and, consequently, $v_1(x, t) \equiv u_1(x, t) + w_1(x, t) \ge \varepsilon/8$ for all $\lambda \in [0, 1]$. We put $s = 8\delta/\varepsilon$. Then, obviously, any trajectory from the initial point in $U_{\delta}(\Gamma_2) \times I_{t_0}$ goes out of $U_{\delta}(\Gamma_2)$, and hence out of Ω after the time $\Delta t \le s$. This, in particular, signifies that in the region $U_{\delta}(\Gamma_2)$ there are no asymptotic trajectories of the flow $\mathbf{v}$.

In a similar manner we consider the part Γ_1; only instead of the flow **v** here it is more convenient to consider the flow $-\mathbf{v}$. The lemma is proved.

Remark 3.1. From the proof of the lemma we note that any trajectory of the flow v exiting from $U_\delta(\Gamma_1)$, enters into Ω_δ, while any trajectory exiting from $U_\delta(\Gamma_2)$, also goes out of Ω.

We now elucidate the behavior of the trajectories of the flow **v** outside the neighborhoods of the parts Γ_1 and Γ_2. We first shall study the properties of the trajectories of the potential flow **u** given by the system of equations

$$z' = \mathbf{u}(z, t). \tag{3.3}$$

The condition (1.6) signifies that the system (3.3) relates to the class of so-called asymptotically autonomous systems, the properties of solutions of which are close to the properties of the solutions of the autonomous system (3.2) [6, 7]. In particular, we can show that in the region Ω there are no asymptotic trajectories of the flow **u** and all trajectories of the flow **u**, starting at $t=0$ on $\bar{\Omega}$, leave $\bar{\Omega}$ after a finite time t_∞.

Indeed, for the flow $\mathbf{u}_\infty$ such time exists and is finite. We denote it by τ_∞. We put $K_2 = \|\nabla \mathbf{u}_\infty\|_{C(\bar{\Omega}_{\delta/2})}$, $\varkappa = \delta/4\tau_\infty \exp(2K_2\tau_\infty)$, and let $t_1 \geq t_0$ be the value of time starting from which $\|\mathbf{u}(x, t) - \mathbf{u}_\infty(x)\|_{C(\bar{\Omega})} < \varkappa$. We take in the region $\bar{\Omega}_\delta$ an arbitrary point x_1 and denote by $x = x(x_1, t_1, \tau)$ and $z = z(x_1, t_1, \tau)$ the solutions of the systems (3.2), (3.3) satisfying the initial condition

$$x|_{\tau=t_1} = z|_{\tau=t_1} = x_1.$$

The difference $z - x$ is obviously the solution of the problem

$$\frac{d}{dt}(z - x) = \mathbf{u}(z, t) - \mathbf{u}_\infty(z) + \mathbf{u}_\infty(z) - \mathbf{u}_\infty(x), \qquad (z - x)|_{\tau=t_1} = 0. \tag{3.4}$$

Scalar multiplying (3.4) by $y - z$ and assuming that $x, z \in \bar{\Omega}_{\delta/2}$, we obtain

$$\frac{d}{dt}|z - x| \leqslant 2K_2|z - x| + 2\varkappa.$$

Hence, applying the lemma of Gronwell, we deduce that at the instant t_2 of exit of the trajectory $x(x_1, t_1, \tau)$ from $\bar{\Omega}_{\delta/2}$ we have

$$|z(x_1, t_1, t_2) - x(x_1, t_1, t_2)| < \delta/2,$$

and, consequently, $z(x_1, t_1, t_2) \in U_\delta(\Gamma_2)$. Since x_1 is an arbitrary point, then, applying Lemma 3.1, which obviously is valid for the flow **u**, we arrive at the sought result. In the role of the time t_∞ we can take the quantity $t_1 + \tau_\infty + 2s$. We formulate the results thus obtained in the form of a lemma.

LEMMA 3.2. There exists a number t_∞ such that any trajectory of the flow **u** from the initial point in $\bar{\Omega} \times I_0$ goes out of Ω after a time interval $\Delta t \leq t_\infty$.

Using Lemma 3.2, we obtain analogous results for the flow **v**. In fact, we put $K_3 = \sup_{0 \leqslant t < \infty} \|\nabla \mathbf{u}\|_{C(\bar{\Omega}_{\delta/2})}$, $\lambda_0 = \delta/4K_1 t_\infty \exp(2K_3 t_\infty)$. Then reasoning in the same way as in the proof of the statement of Lemma 3.2, we arrive at the following result.

LEMMA 3.3. There exist numbers $\lambda_0 > 0$ and S such that for all $\lambda \in [0, \lambda_0]$ any trajectory of the flow **v** from the initial point in $\bar{\Omega}_\delta \times I_{t_0}$ goes out of Ω_δ into the region $U_\delta(\Gamma_2)$ after a time interval $\Delta t \leq S$.

We put $T_\infty = t_0 + 2s + S$. Then from Lemmas 3.1 and 3.3 in an obvious manner we have the following theorem.

THEOREM 3.1. Let the conditions (1.6), (3.1) be fulfilled. Then there exist numbers λ_0 and T_∞ such that for all $\lambda \in [0, \lambda_0]$ in the region Ω there are no asymptotic trajectories of the flow **v**, with any trajectory of the flow v commencing at $t=0$ on Ω, going out of Ω at the instant $t \leq T_\infty$.

From Theorems 2.1 and 3.1 follows the basic theorem.

THEOREM 3.2. Let the conditions I-IV be fulfilled. Then there exist numbers $\lambda_0 > 0$ and T_∞ such that for all $\lambda \in [0, \lambda_0]$ and $t \geq T_\infty$ rot $\mathbf{v} \equiv 0$ and, consequently, $\mathbf{v}(x, t) \equiv \mathbf{u}(x, t)$.

Theorem 3.2 has two obvious corollaries.

COROLLARY 1. (Theorem of Settling). Under the conditions of Theorem 3.2 the solution of the problem (1.1) for all $\lambda \in [0, \lambda_0]$ uniformly tends, for $t \to \infty$, to the potential flow $\mathbf{u}_\infty(x)$ as the only solution of the corresponding stationary problem (1.2). In addition, if $\gamma(x, t) \equiv \gamma_\infty(x)$ for $t \geq T_\infty$, then the solution of the problem (1.1) settles relative to the solution of the problem (1.2) after a finite time T_∞.

COROLLARY 2. (Theorem on Asymptotic Stability of a Potential Flow). Under the conditions of Theorem 3.2 the potential flow $\mathbf{u}_\infty(x)$ is asymptotically stable relative to small perturbations which are potential at the entry of the region.

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INCLINATION ANGLES OF THE BOUNDARY IN MOVING LIQUID LAYERS

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In this paper creeping flows in thin layers of a viscous liquid are discussed with the capillary forces taken into account, and solutions describing the inclination angles of the boundary are found. The contact angle of a liquid on a solid surface in the static state is expressed in terms of the specific surface energies. Upon movement of the liquid the contact angle (dynamic) differs from the static value. A very thin "precursor" film can be observed in front of the liquid mass which is spreading over the solid surface [1, 2]. There are indications to the effect that the value of the dynamic contact angle depends on the viscous forces [3].

1. Established Flow of a Liquid Layer over a Dry Surface and the Contact Angles. The pressure p inside a thin liquid layer on a flat solid surface differs from the pressure p_0 in the gas by the amount of the capillary differential $p = p_0 - \sigma \partial^2 h/\partial x^2$ (σ is the surface tension coefficient; x is the coordinate along the layer; and h is the thickness of the layer).

The equation of motion of the layer in the case of small Reynolds numbers under the action of capillary forces can be written with the help of the hydrodynamical theory of lubrication as

$$\frac{\partial}{\partial x}\left(\frac{\sigma}{3\mu} h^3 \frac{\partial^3 h}{\partial x^3}\right) = -\frac{\partial h}{\partial t}.$$

Non-steady-state solutions of this equation are investigated in the linear approximation in [4]. Let us consider steady-state solutions in the nonlinear formulation. For a steady-state wave

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